An Alternative Method to Estimate Balancing Factors for the Disaggregation of OD Matrices

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1   **ABSTRACT**
2   The solution algorithms for the family of flow distribution problems, which include (1)
3   the trip distribution problem of travel forecasting, (2) the OD estimation from link counts
4   problem, and (3) the trip matrix disaggregation problem, are usually based on the Maximum
5   Entropy (ME) principle. ME-based optimization problems are hard to solve directly by
6   optimization techniques due to the complexity of the objective function. Thus, in practice,
7   iterative procedures are used to find approximate solutions. These procedures, however, cannot
8   be easily applied if additional constraints are needed to be included in the problem.
9   In this paper a new approach for balancing trip matrices with application in trip matrix
10   disaggregation is introduced. The concept of generating the most similar distribution (MSD)
11   instead of the Most Probable Distribution of Maximum Entropy principle is the basis of this
12   approach. The goal of MSD is to minimize the deviation from the initial trip distribution, while
13   satisfying additional constraints. This concept can be formulated in different ways. Two MSD-
14   based objective functions have been introduced in this paper to replace the ME-based objective
15   function. One is the Sum of Squared Deviations MSD (SSD-MSD), and the other is Minimax-
16   MSD. While SSD-MSD puts more emphasis on maintaining the base year trip shares as a whole,
17   Minimax-MSD puts more emphasis on maintaining the share of each individual element in the
18   trip table.
19   The main advantage of replacing the entropy-based objective functions with any of these
20   functions is that the resulting problems can include additional constraints and still be readily
21   solved by standard optimization engines. In addition, these objective functions could produce
22   more meaningful results than entropy-based functions in regional transportation planning studies,
23   as shown in the case study and some of the examples in the paper. Several examples and a case
24   study of the California Statewide Freight Forecasting Model (CSFFM) are presented to
25   demonstrate the merits of using MSD-based formulations.
26
27   **Keywords:** trip distribution, maximum entropy, trip matrix disaggregation, Most Similar
28   Distribution
29
30

INTRODUCTION

The trip distribution problem is a familiar component of passenger and freight transportation planning models. It may appear in the form of trip matrix disaggregation for sub-region analysis, trip table generation based on traffic counts, or trip matrix balancing to match forecasts of zonal productions and attractions. The most widely used methods in each of these cases are directly or implicitly based on Maximum Entropy (ME) theory. This principle implies that in the absence of prior information, the probability distribution which best represents the current state of knowledge is the one with the largest entropy. In other words, if there are no preferences among various outcomes, then all the possible outcomes are assumed to have uniform probability *(1)*.

ME-based formulations are too complex to be directly solved by optimization engines. Multiple iterative methods founded on ME theory have been developed to solve different forms of the trip distribution problem and have been widely used due to their ease of computation and fast convergence rate. However, the restrictive framework of the iterative procedures doesn't allow an exhaustive use of the information in hand.

Given the significant increase in computational speeds and wide-spread availability of powerful open source optimization packages, the long-held "benefits" of ME-based methods no longer outweigh their possible errors in final trip distribution results. It is time to evaluate the impacts of using ME-based methods on final trip distribution results and potentially use alternative methods that make use of all the available accessory data sources. This area of research has received little attention to the best of the authors' knowledge.

The goal of this paper is to investigate alternative methods by which to approach the trip distribution problem, with a focus on its application in freight transportation. Two alternative models are proposed to estimate or adjust trip tables in situations where ME-based methods may not be applicable or provide valid results. The proposed models can be easily handled by available optimization packages, include modal accessibility constraints, and maintain consistency with aggregate observed trip tables while also minimizing deviations from original trip shares when estimating balancing factors for future trip tables.

We believe that the theoretical arguments made in this paper and the practical significance of using the alternative models combined implies that these models could be suitable to be used in practice where appropriate.

In the next section, methods used to solve distribution models with a focus on ME methods are reviewed. This is followed by an examination of the impacts of ME assumptions on trip table estimation including example cases and presentation of proposed models. Finally, the California Statewide Freight Forecasting Model (CSFFM) is presented as an important application of the proposed methods.

LITERATURE REVIEW

In the transportation literature, the term distribution models generally refers to the methods used in the second step of the Four Step Model for travel forecasting, where the total number of trips beginning and ending in each zone is known and the goal is to find the number of trips between each pair of zones. In a matrix representation, the objective is to populate the cells of a matrix given row and column marginal values.

In this paper, this family of models is extended to include not only those estimated from marginal values of the matrix itself, but also those methods that use any other available information at any level for estimation, that is both estimating at coarser levels based on detailed

1   data, as in traffic OD estimation, or estimating at finer levels based on aggregate data as in
2   matrix disaggregation. Based on this definition, the "matrix disaggregation problem" and the
3   "OD estimation problem" can be categorized as distribution problems.
4   ME-based models have been the method of choice when working with limited
5   information to find the most likely solution amongst many alternates and have been widely used
6   in distribution models. These three defined problems deemed the "distribution problem" will be
7   presented in detail with an emphasis on the solution methods that use ME.
8   The first problem is the estimation of a trip table where forecast productions and
9   attractions and a base year trip table are available. Each zone may have two different growth
10   factors for in- and out-bound trips. To obtain the future trip table a number of iterative methods
11   have been proposed. The most well-known of these methods were developed by (and named
12   after) Furness *(2)* and Fratar *(3)*. It was later shown that the Furness model can be derived from a
13   doubly constrained optimization problem using ME as the objective function.
14   Gravity models are one of the first theory-based methods to be used for trip distribution
15   modeling. Assuming that the total productions and consumptions by all the zones in the network
16   are known, gravity models distribute these marginal values in trip tables estimating the most
17   likely number of trips between each origin and destination, given known impedance values for
18   each OD pair. Wilson *(4)* derived the classic gravity model from the Lagrangian form of an
19   optimization problem using the ME objective function and production and attraction constraints.
20   A number of authors have attempted to formulate more general versions of the classical gravity
21   model. For example, Fang and Tsao *(5)* generalized the classical gravity model to a "self-
22   deterrent" gravity model by adding a "congestion term" to the exponent in the formulation of
23   gravity model. Additionally, De Grange et al. *(6)* included a term in the exponent to account for
24   accessibility to destinations.
25   Another theoretical model for trip distribution is the intervening opportunities model.
26   This model was initially proposed by Stouffer *(7)*, but the theoretical formulation known today
27   was developed by Schneider *(8)*. The basic idea behind the intervening opportunities model is
28   that destination choice is not explicitly related to distance, but rather to the accessibility to
29   opportunities that can satisfy the trip's purpose. The main difference between the intervening
30   opportunities model and the gravity model is in the use of the distance variable. While in the
31   latter model distance is a continuous variable, it is treated as an ordinal variable in the former
32   model. The intervening opportunities model is not widely used by practitioners, due to the less
33   well known and more complicated theoretical basis, and lack of suitable software. Wilson *(4)*
34   also showed that the intervening opportunities model can be derived from an ME-based
35   formulation.
36   While aggregate distribution models are estimated at the zonal level, disaggregate
37   approaches are estimated at the individual household level. As opposed to aggregate approaches
38   where the output of the distribution model is the number of trips between zones, disaggregate
39   methods estimate the probability that an individual chooses a specific destination to satisfy the
40   need for an activity. Discrete choices models are usually used for the purpose of disaggregate
41   estimation of trip distribution *(9)*. Wilson *(10)* also showed that multinomial logit models can be
42   derived from an ME-based optimization.
43   The next problem is matrix disaggregation. In transportation modeling, fine disaggregate
44   level data can be expensive or impractical to obtain, so many transportation models are
45   calibrated at coarser, aggregate levels and then disaggregated using local data. The limited
46   availability of disaggregate data is a more significant issue in freight transportation due to the

1   proprietary nature private sector data that is of interest in freight modeling. The Freight Analysis
2   Framework (FAF) *(11)* is the only national publicly available commodity flow database.
3   However, FAF zones are very aggregate (there are only 123 domestic regions in the last version
4   of FAF).
5   Several attempts have been made to disaggregate FAF flows to county or sub-county
6   levels. For instance, Alam and Majed *(12)* proposed a methodology to disaggregate the FAF
7   Truck OD matrices to the county level using proportional allocation techniques based on land
8   use or employment data at the county level and total truck VMT within that county to facilitate
9   freight assignment on interstate and state highways. Opie et al.*(13)* developed several weighting
10   schemes to disaggregate FAF flows proportionally. The recommended scheme was based on
11   NAICS 3-digit employment *(14)*. They also used a tri-proportional method to balance the final
12   disaggregate matrix. Cambridge Systematics *(15)* and Viswanathan et al. *(16)* proposed similar
13   methodologies to divide FAF2 regional commodity flow data for all commodities into county
14   level flows by commodity. They developed robust linear regression equations to generate total
15   production and attraction for each county. These regressions were used to guide the development
16   of factors for each commodity for the disaggregation of freight flow productions and attractions.

17   **ALTERNATIVE MODEL FOR THE TRIP DISTRIBUTION PROBLEM**
18   Two alternative models are proposed to replace the ME-based models that serve as the
19   basis of the three types of distribution problems introduced above. These alternative methods
20   will be of interest especially when other data regarding capacity, constraints on transportation
21   facilities, or aggregate trip matrices are available. However, since the case study in this paper is
22   about trip distribution and disaggregation, these example models are presented accordingly.

23   **Maximum Entropy (ME) Model for Matrix Balancing**
24   The iterative process proposed by Furness *(2)*, also known as the bi-proportional method,
25   has been very popular and, in fact, is included in the majority of transportation software
26   packages as the standard procedure by which to estimate balancing factors for trip distribution
27   matrices. Similar to doubly constrained growth factor models, the bi-proportional method
28   requires target production ($P_i \forall i \in Z$) and target attraction ($A_j \forall j \in Z$) estimates, and an initial
29   trip table (t_{ij}), where Z is the set of zones in the networks.
30   The iterative solution process produces an approximate solution based on the Lagrangian
31   of an optimization problem known as the "maximum entropy transportation model" or the
32   "maximum entropy special interaction model", presented in Eqs.1.1 to 1.4:
33

$$\text{Min } G_{ME} = \Sigma_{ij} T_{ij} \left(\log \frac{T_{ij}}{t_{ij}} - 1 \right) \qquad \text{Eq. 1.1}$$

Subject to

$$\Sigma_j T_{ij} = P_i \ \forall i \in Z \qquad \text{Eq. 1.2}$$
$$\Sigma_i T_{ij} = A_j \ \forall j \in Z \qquad \text{Eq. 1.3}$$
$$T_{ij} \geq 0 \ \forall \text{i}, \text{j} \in Z \qquad \text{Eq. 1.4}$$

34
35   Although this iterative fitting method has been used widely in transportation, in some
36   cases there is merit in replacing it with other methods.

1   **Most Similar Distribution (MSD) Model**
2   In multi-regional transportation analysis, it is common to assume that the entire system is
3   stable and any relationships, transactions, or flows among regions are in equilibrium. This
4   concept can be applied to trip distribution by assuming that future trip shares are the same or
5   close to present trip shares (the shares $[s_{ij}]$ for a trip table $[t_{ij}]$ can be obtained by normalizing
6   the trip table to 1, i.e. by dividing the elements of the table by the sum of the table elements, t_{ij}).
7   Under this formulation, instead of maximizing entropy as is the case for ME-based methods, the
8   goal is to minimize the difference between the $[T_{ij}]$ and $[t_{ij}]$ trip shares, that is, to generate the
9   MSD given the production and attraction constraints. The objective function in MSD-based trip
10   distribution problem can be formulated in different ways depending on the goals of the
11   researcher and attributes of initial trip distribution. In the following section two formulations are
12   presented and compared with several examples.

13   *Sum of Squared Deviations MSD (SSD-MSD)*
14   The first proposed formulation is to minimize sum of the squared deviations between the
15   estimated future year and base year trip shares. The model formulated for trip distribution in
16   freight and passenger models is presented in Eqs. 2.1 to 2.5.
17

$$\text{Min } G_{SSD-MSD} = \Sigma_{ij}\left(\delta_{ij}\right)^2 \qquad \text{Eq. 2.1}$$

Subject to:

$$\Sigma_j T_{ij} = P_i \; \forall i \in Z \qquad \text{Eq. 2.2}$$

$$\Sigma_i T_{ij} = A_j \; \forall j \in Z \qquad \text{Eq. 2.3}$$

$$\delta_{ij} = \frac{T_{ij}}{\Sigma_{ij}T_{ij}} - \frac{t_{ij}}{\Sigma_{ij}t_{ij}} \; \forall i,j \in Z \qquad \text{Eq. 2.4}$$

$$T_{ij} \geq 0 \; \forall i,j \in Z \qquad \text{Eq. 2.5}$$

18
19   The T_{ij} terms in the above formulation are the decision variables and represent the trip
20   table in the forecast year. Eqs.2.2 and 2.3 ensure that the forecast year production and attraction
21   constraints are satisfied. Clearly in order for the problem to be feasible, total production should
22   be equal to total attraction (i.e. $\Sigma_i P_i = \Sigma_j A_j$). The δ_{ij} variables measure the difference between
23   the base and forecasted trip shares. The objective function of the SSD-MSD problem attempts to
24   minimize sum of the squared deviations between the estimated future year and base year trip
25   shares.
26   If constraints 2.3 and 2.4 allow, then the objective function (2.1) is minimized when the
27   δ_{ij} terms are all equal in value. So the same change in share of two cells in the matrix, say
28   δ_{ij}=0.01, could have different implications depending on the base year shares. For example, if
29   the base year share is 0.1, δ_{ij}=0.01 means that the forecast year share is 0.11. Similarly, for a
30   base year share of 0.01, a δ_{ij}=0.01 means that the future year share is 0.02. In the first case, the
31   share has changed from 0.1 in the base year to 0.11 in the forecast year, a change of only 10
32   percent. However, in the second case, the share has changed from 0.01 in the base year to 0.02 in
33   the forecast year, a change of 100 percent. This is an advantage for regional analysis where zones
34   with small volumes of flow between them are expected to have more fluctuation in volumes of
35   flow in the future, compared to zones that already have strong and stable flow between them.
36   One advantage of having a simple mathematical form of an objective function is that the
37   constraints of the problem are not restricted to production and consumption constraints. Any

1   capacity constraint that can be formulated as a linear function of decision variables or any
2   relationship between flows of different OD pairs can be easily added to the problem, without
3   having a considerable impact on the time complexity of reaching the optimal solution. Although
4   such constraints can also be added to more complicated non-linear objective functions like ME,
5   solving such non-linear problems via standard optimization packages is not as easy. Usually to
6   solve such non-linear problems, algorithms are developed based on the Lagrangian form of the
7   problem and specific optimization techniques such as column generation are used to solve the
8   resulting sub-problems.
9   In the next section, a simple numerical example is presented to compare SSD-MSD and
10 ME closely. The other possible and desired formulation for objective function will be discussed
11 later in the paper.

12 **Example 1: Comparison of SSD-MSD and ME** Assume that there exists one origin and two
13 destination zones in a network, and the initial value of each cell in the trip table is $(t_{11}, t_{12}) =$
14 $(0.5, 0.5)$, in units of hundred trips (half the production of zone 1 goes to zone 2 and the other
15 half is intra-zonal flow). Also, for simplicity, the only constraints applied are the non-negativity
16 constraints. The surfaces of the solution space of the SSD-MSD and ME models are displayed in
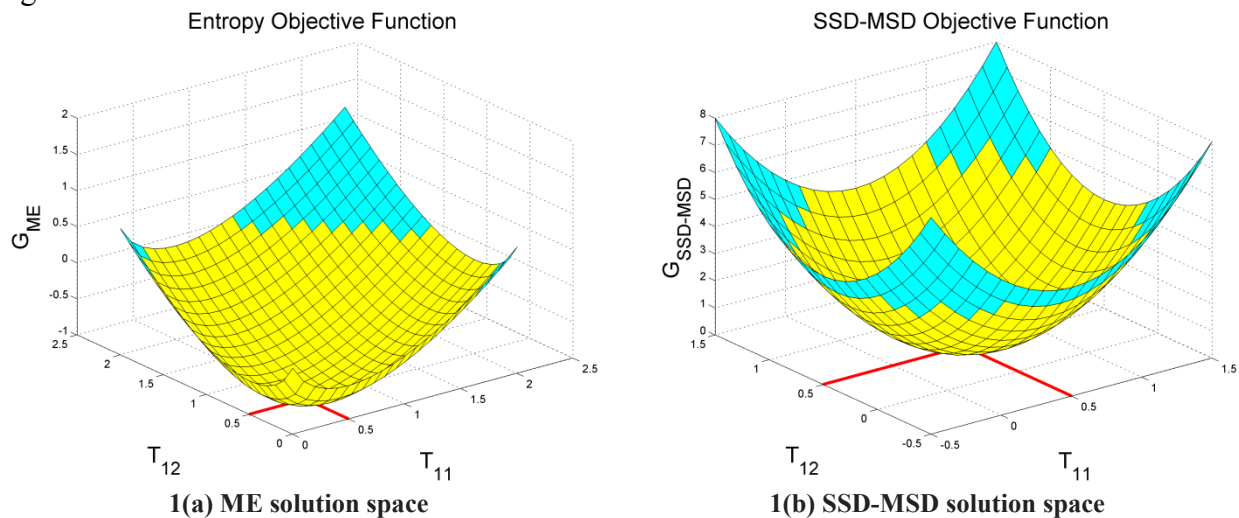17 Figure 1.

1(a) ME solution space   1(b) SSD-MSD solution space

FIGURE1 Comparing the ME and SDS-MSD solution space

18   As shown in Figure 1(b), the second derivative of the SSD-MSD function is a constant,
19 and therefore $G_{SSD-MSD}$ is a symmetric function and its curvature doesn't depend on the value of
20 the decision variables, while the curvature of G_{ME} changes with the decision variables. Clearly in
21 this case both models are minimized at $(T_{11}, T_{12}) = (0.5, 0.5)$, returning the initial trip shares.
22   Expanding the example further, the effect of adding constraints to both optimization
23 problems will be analyzed next. The first constraint added to the problems is $T_{11} + T_{12} \leq 0.5$.
24 Figures 2(a) and 2(b) show the contours of the two objective functions, G_{ME} and $G_{SSD-MSD}$
25 respectively, with the added constraint. Both problems are minimized at $(T_{11}, T_{12}) = (0.25, 0.25)$
26 (which can be normalized to trip shares of $(0.5, 0.5)$, maintaining the original trip shares.
27   Now consider the initial scenario, but with different base year trip table elements
28 $(t_{11}, t_{12}) = (0.8, 0.2)$. In this scenario, the constraint $T_{11} + 0.29\, T_{12} \geq 1$ is added to both
29 problems.

1   In this case, the two problems do not return the same optimal solution. The dotted blue
2   lines in Figures 2(c) and 2(d) show the initial trip rates, and the dotted red lines show the optimal
3   solutions after adding the constraint. The optimal solution by the SSD-MSD model occurs at
4   (1.92,0.44), which can be normalized to trip shares of (0.81,0.19), very close to the initial trip
5   share,(0.8,0.2). The ME model, however, changes the initial trip share from (0.8,0.2) to
6   (0.57,0.43). In fact, the entropy function leans toward a uniform distribution of changes in
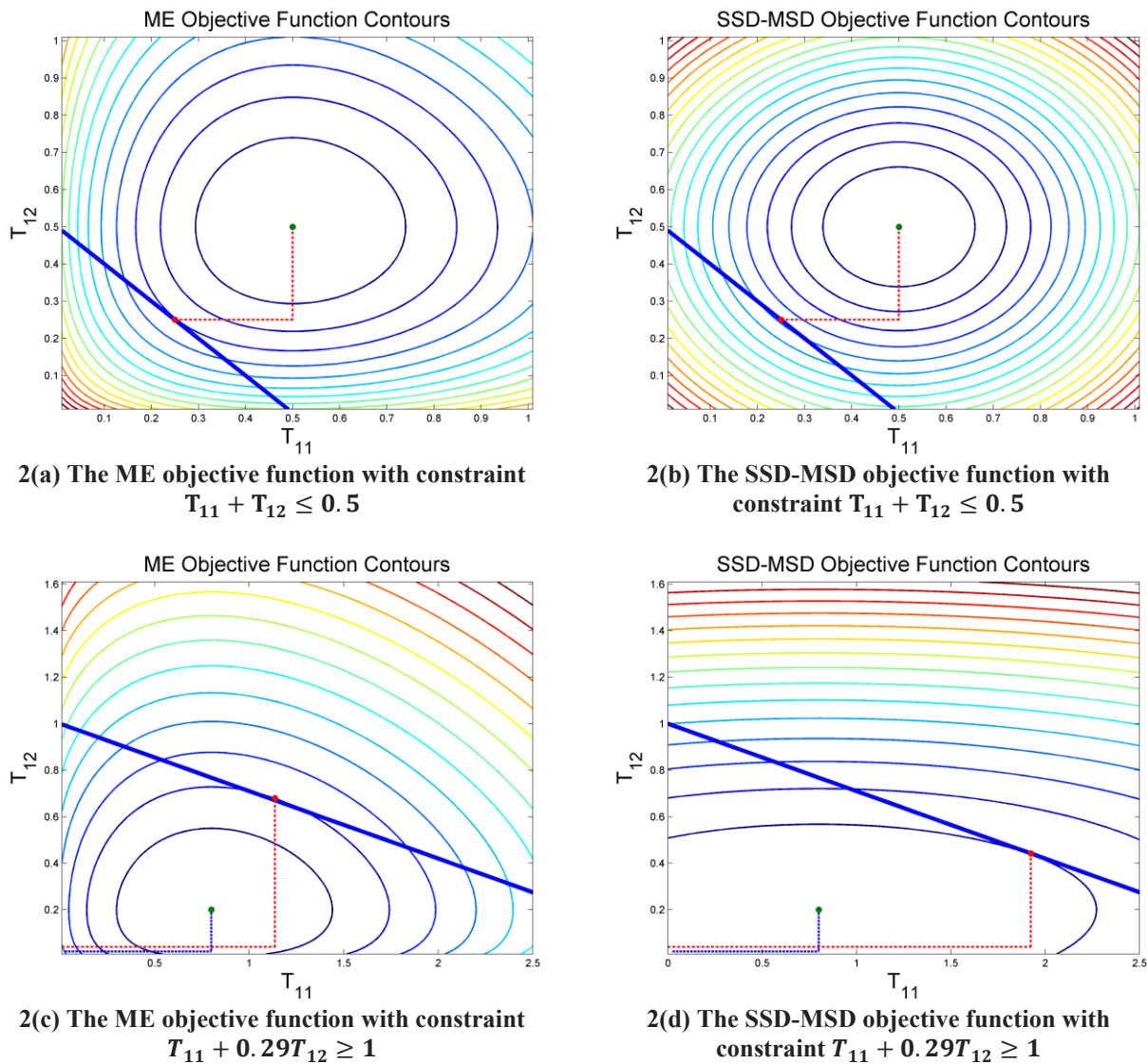7   number of trips.
8

2(a) The ME objective function with constraint $T_{11} + T_{12} \leq 0.5$

2(b) The SSD-MSD objective function with constraint $T_{11} + T_{12} \leq 0.5$

2(c) The ME objective function with constraint $T_{11} + 0.29T_{12} \geq 1$

2(d) The SSD-MSD objective function with constraint $T_{11} + 0.29T_{12} \geq 1$

FIGURE2 Example 1: contours of ME and SSD-MSD objective functions under different constraints

9   *Minimax-MSD*
10   In SSD-MSD, the sum of the squared deviations from the initial trip shares is minimized.
11   This implies that some cells may experience large deviations while other cells experience little or
12   no deviation. A less discriminating application of the same principle is performed using the
13   Minimax objective function. The objective function of this problem minimizes the maximum

1 deviation of the resulting trip shares from the base year trip shares in the form: $MinMax\ |\delta_{ij}|$.
2 Since an absolute value cannot be handled directly by linear optimization software, mathematical
3 transformations are needed. The transformed form of the problem is presented in Eq. 3.1 to 3.7.
4
5

$$\text{Min } G_{Minimax\_MSD} = G_{MM} \hspace{3cm} \text{Eq. 3.1}$$
$$\text{Subject to}$$
$$\Sigma_j T_{ij} = P_i\ \forall i \in Z \hspace{3cm} \text{Eq. 3.2}$$
$$\Sigma_i T_{ij} = A_j\ \forall j \in Z \hspace{3cm} \text{Eq. 3.3}$$
$$\delta_{ij} = \frac{T_{ij}}{\Sigma_{ij}T_{ij}} - \frac{t_{ij}}{\Sigma_{ij}t_{ij}}\ \forall i,j \in Z \hspace{2cm} \text{Eq. 3.4}$$
$$\delta_{ij} = \delta_{ij}^+ - \delta_{ij}^-\ \forall i,j \in Z \hspace{2.5cm} \text{Eq. 3.5}$$
$$G_{MM} \geq \delta_{ij}^+ + \delta_{ij}^-\ \forall i,j \in Z \hspace{2.2cm} \text{Eq. 3.6}$$
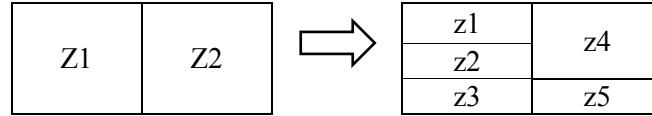$$T_{ij}, \delta_{ij}^+, \delta_{ij}^- \geq 0\ \forall i,j \in Z \hspace{2.3cm} \text{Eq. 3.7}$$

6
7   Eqs. 3.2 and 3.3 are the production and consumption constraints. Eq. 3.4 defines δ_{ij} as
8 the difference between the base year and forecast year trip shares. Eq. 3.5 sets δ_{ij}, to be the
9 difference between two positive variables, δ_{ij}^+ and δ_{ij}^-. This transformation is required to turn the
10 objective function $MinMax\ |\delta_{ij}|$ into the form $x\left(\delta_{ij}^+ + \delta_{ij}^-\right)$. Eq. 3.6 is added to further
11 simplify the objective function. The term $Max\left(\delta_{ij}^+ + \delta_{ij}^-\right)$ in the objective function is replaced
12 by G_{MM}, and Eq. 3.6 is added to ensure that G_{MM} represents the max value of $\left(\delta_{ij}^+ + \delta_{ij}^-\right)$s over
13 all the i and j indices. Lastly, Eq. 3.7 enforces the non-negativity constraints.

14 **Example 2: Application of MSD-based models in OD disaggregation** Assume that the
15 aggregate trip table for a study area comprising two zones (Z1 and Z2) is available and that a
16 disaggregate sub-regional analysis of the network is required. Assume Z1 is to be divided into
17 three zones (z1, z2, and z3) and Z2 is to be divided into two zones (z4 and z5) as displayed in
18 Figure 3(a). A trip table of the disaggregate network with five zones is available from previous
19 sub-regional studies, and there is reason to believe that the trip shares have not changed in the
20 intervening time period. Therefore, this trip table represents the base year trip table for an MSD-
21 based optimization problem. The aggregate trip table for Z1 and Z2 and the base year
22 disaggregate trip table for z1, z2, … , z5 are presented in Figures 3(b) and 3(c).
23   The goal is to readjust the disaggregate trip table such that it complies with the
24 corresponding aggregate trip table, while disturbing the base year disaggregated trip shares as
25 little as possible. Towards this end, the SSD-MSD problem is formulated in Eqs. 4.1 to 4.7.

3(a) Update the zoning system in the network

Zone	1	2	Prod.
1	10	7	17
2	8	6	14
Attr.	18	13	31

3(b)observed aggregated trip table

Zone	1	2	3	4	5	Prod.
1	1	3	3	3	1	11
2	1	2	5	2	5	15
3	4	1	3	3	4	15
4	3	2	1	5	1	12
5	4	3	6	3	3	19
Attr.	13	11	18	16	14	72

3(c) Base year disaggregated trip table

FIGURE 3.Matrix disaggregation example. Color coded aggregate and disaggregate matrices

1

$$Min\ G_{SSD-MSD} = \sum_{i=1}^{5}\sum_{j=1}^{5} \delta_{ij}^2 \qquad\qquad \text{Eq. 4.1}$$

Subject to:

$$\sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij} = 10 \qquad\qquad \text{Eq. 4.2}$$

$$\sum_{i=1}^{3}\sum_{j=4}^{5} T_{ij} = 7 \qquad\qquad \text{Eq. 4.3}$$

$$\sum_{i=4}^{5}\sum_{j=1}^{3} T_{ij} = 8 \qquad\qquad \text{Eq. 4.4}$$

$$\sum_{i=4}^{5}\sum_{j=4}^{5} T_{ij} = 6 \qquad\qquad \text{Eq. 4.5}$$

$$\frac{T_{ij}}{\sigma_z} + \delta_{ij} = \frac{t_{ij}}{\sigma_z}, \quad i,j = 1,2,3,4,5 \qquad\qquad \text{Eq. 4.6}$$

$$T_{ij} \geq 0, \quad i,j = 1,2,3,4,5 \qquad\qquad \text{Eq. 4.7}$$

2
3   In Eq. 4.7, $\sigma_z = 31$, the sum of the observed trips for Figure 3(b), and $\sigma_z = 72$, the sum
4   of the base trips in Figure 3(c).
5   The solution is presented in Figure 4(a). Figures 4(b) and 4(c) display the trip shares of
6   the base and current years, respectively. δ_{ij}s, the difference between these two shares, are
7   presented in Figure 4(d). The Minimax-MSD formulation for this problem includes constraint
8   Eqs.4.2 through 4.7 along with the additional constraints in Eqs.5.2, 5.3, and 5.4 shown below.

$$\text{Eq. 5.1}$$
$$Min\ G_{Minimax-MSD} = Z$$

Subject to:

$$Z \geq \delta_{ij}^+ + \delta_{ij}^- ; \quad i,j = 1,2,3,4,5 \qquad\qquad \text{Eq. 5.2}$$
$$\delta_{ij} = \delta_{ij}^+ - \delta_{ij}^-; \quad i,j = 1,2,3,4,5 \qquad\qquad \text{Eq. 5.3}$$
$$\delta_{ij}^+, \delta_{ij}^- \geq 0; \quad i,j = 1,2,3 \qquad\qquad \text{Eq. 5.4}$$

Eqs. 4.2-4.7

1   The optimal values of δ_{ij} from solving problem 5 are presented in Figure 4(e). Figure 4
2 shows that the maximum change in the optimal trip shares generated by Minimax-MSD equals
3 0.0067, while the maximum change in trip share generated by the SSD-MSD is 0.074. This
4 shows the primary merit of the Minimax-MSD objective function: all cells carry equal weight.

Zone	1	2	3	4	5	Production
1	0.44	1.30	1.30	1.17	0.31	4.52
2	0.44	0.87	2.16	0.74	2.03	6.24
3	1.73	0.44	1.30	1.17	1.60	6.24
4	1.26	0.83	0.40	2.38	0.62	5.49
5	1.69	1.26	2.55	1.50	1.50	8.51
Attraction	5.57	4.71	7.72	6.95	6.05	31.00

4(a) solution: current year trip table by SSD-MSD

Zone	1	2	3	4	5
1	0.014	0.042	0.042	0.042	0.014
2	0.014	0.028	0.069	0.028	0.069
3	0.056	0.014	0.042	0.042	0.056
4	0.042	0.028	0.014	0.069	0.014
5	0.056	0.042	0.083	0.042	0.042

4(b) base year trip shares

Zone	1	2	3	4	5
1	0.014	0.042	0.042	0.038	0.010
2	0.014	0.028	0.070	0.024	0.065
3	0.056	0.014	0.042	0.038	0.052
4	0.041	0.027	0.013	0.077	0.020
5	0.055	0.041	0.082	0.048	0.048

4(c) current year trip shares by SSD-MSD

Zone	1	2	3	4	5
1	0.0003	0.0003	0.0003	-0.0040	-0.0040
2	0.0003	0.0003	0.0003	-0.0040	-0.0040
3	0.0003	0.0003	0.0003	-0.0040	-0.0040
4	-0.0010	-0.0010	-0.0010	0.0074	0.0061
5	-0.0010	-0.0010	-0.0010	0.0067	0.0067

4(d) Optimal δ_{ij}'s by SSD-MSD

Zone	1	2	3	4	5
1	0.0031	0.0067	0.0067	-0.0040	0.0000
2	0.0067	0.0067	-0.0067	-0.0067	-0.0067
3	-0.0067	-0.0067	-0.0067	-0.0067	0.0000
4	0.0000	0.0009	0.0000	0.0067	0.0067
5	-0.0067	-0.0067	0.0067	0.0067	0.0067

4(e) Optimal δ_{ij}'s by Minimax-MSD

FIGURE 4 Comparison of solutions to the SSD-MSD and Minimax-MSD problems

5 **CASE STUDY: CALIFORNIA STATEWIDE FREIGHT FLOW MATRIX**
6   The MSD model was used in the California Statewide Freight Forecasting Model
7 (CSFFM) as the only viable method that can address all the modeling concerns of its developers.
8 CSFFM is a commodity based interregional freight transportation model which focuses on rail
9 and truck movements. It is based on the FAF3 public data source *(11)*. The model covers 215
10 Freight Analysis Zones (FAZ) including 96 zones in California and 118 zones outside California
11 in the United States. In less populated areas, FAZs are represented by counties and in the more
12 developed central and southern parts of California, FAZs are sub-counties (Figure 5). There are
13 also 15 commodity groups in the model. Details of data preparation and the commodity
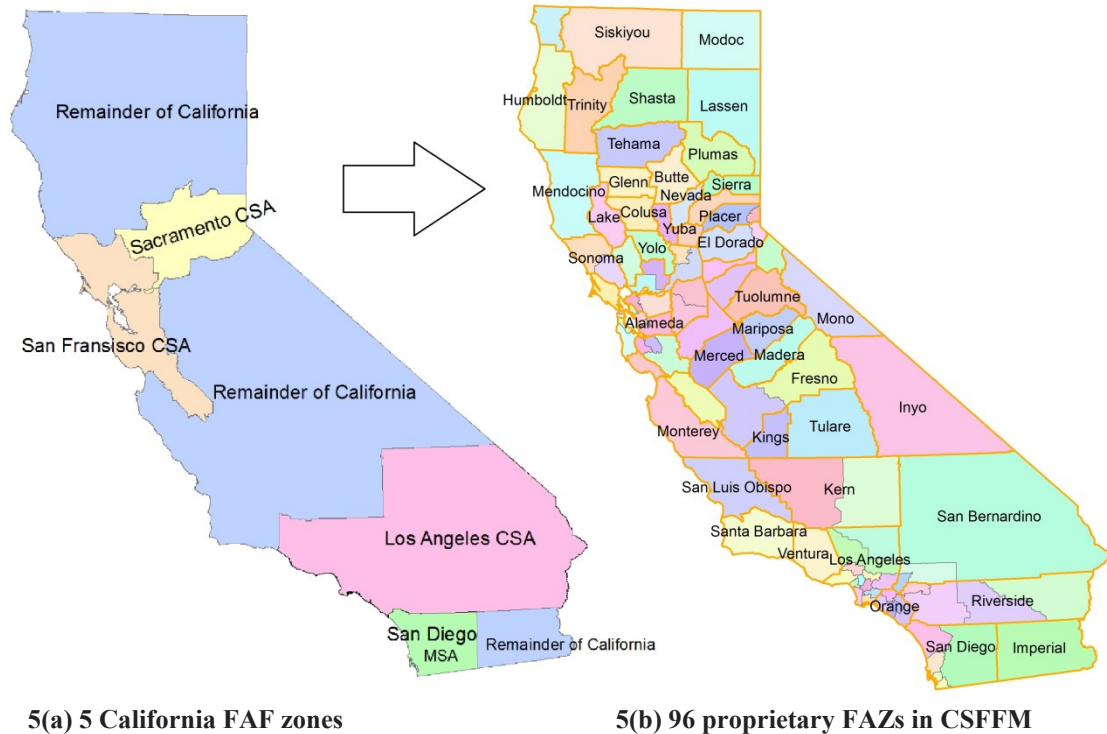14 generation process are provided in Ranaiefar et al. *(17)*.
15

5(a) 5 California FAF zones   **5(b) 96 proprietary FAZs in CSFFM**

FIGURE 5. CSFFM Zoning System

1   CSFFM has a hybrid generation/distribution model. The goal of this model is to
2 disaggregate FAF flows from five FAF regions in California to 97 FAZs for the base year and
3 then to forecast flows for years 2020 and 2040 from base year estimations and scenario
4 assumptions. Initially, in the total generation model the total production and attraction of each
5 zone for all commodities including imports and exports were estimated simultaneously using
6 Structural Equations Modeling (SEM) *(17, 18)*. Next, a domestic commodity flow matrix was
7 estimated by a SEMCOD Model (Structural Equations for Multi-Commodity OD Distribution).
8 SEMCOD is a spatial interaction econometric model, also known in the transportation literature
9 as a direct demand model, which combines trip generation and distribution steps in conventional
10 four-step demand models. It estimates an initial flow matrix for the base year for all commodity
11 groups simultaneously using SEM techniques. SEMCOD uses demographic, economic, land-use,
12 and composite impedances between zones to estimate an initial 215 by 215 OD matrix for each
13 commodity group. The difference between marginals of domestic flow matrices and total
14 productions and attractions estimated by the total generation model is assumed to represent total
15 exports and imports, respectively.
16   SEMCOD and total generation models are estimated independently. An additional
17 procedure is needed to synchronize these models to (1) ensure consistency with the aggregate
18 FAF domestic flow matrices and total imports and exports for FAF geographic regions; (2)
19 ensure that marginals of domestic commodity flow matrices are less than total production and
20 attraction estimated by the total generation model, and (3) ensure modal accessibility and
21 capacity constraints are not violated in disaggregated flow matrices. A customized SSD-MSD
22 model is used to achieve these goals.
23   FAF is a multi-dimensional commodity flow matrix (one matrix for each commodity
24 group). These matrices are very sparse. On average 67% of cells are zero or the flow volume is

1 less than 1 k-ton/year for each commodity group. Cells with very small flows are subject to
2 greater measurement error and sampling bias. Also intra-zonal flows account for more than 50%
3 of the total flows for each commodity group. Given these properties of FAF commodity flow
4 matrices and available information on modal facilities in each FAZ, using a Furness process to
5 estimate balancing factors is not suitable. The SSD-MSD model presented in Eq. 6 considers all
6 the constraints while minimizing the deviation from initial domestic commodity distributions
7 estimated by the SEMCOD model.
8   Other data sources used in this study included a detailed California rail carload waybill
9 sample *(19)*, seaport data from the US Army Corps of Engineers waterborne database *(20)* and
10 land port data from the North American Transborder Freight Data *(21)*.
11

$$Min \sum_{i,j} B^2 (\delta_{ij}^z)^2 + \sum_i (\delta_i^P)^2 + \sum_j (\delta_j^A)^2 \qquad \text{Eq. 6.1}$$

Subject to:

$$\sum_j T_{ij} = P_i - Exp_i \forall i \in z \qquad \text{Eq. 6.2}$$

$$\sum_i T_{ij} = A_j - Imp_j \forall i \in z \qquad \text{Eq. 6.3}$$

$$\sum_{\substack{i, \in I \\ j \in J}} T_{ij} = F_{IJ} \forall I, J \in Z \qquad \text{Eq. 6.4}$$

$$\sum_{i \in I} P_i = Prod_I^F \forall I \in Z \qquad \text{Eq. 6.5}$$

$$\sum_{j \in J} P_j = Attr_J^F \forall J \in Z \qquad \text{Eq. 6.6}$$

$$\frac{t_{ij}}{\sum_{i,j} t_{ij}} + \delta^z_{ij} = \frac{T_{ij}}{\sum_{I,J} F_{IJ}} \forall i, j \in z \qquad \text{Eq. 6.7}$$

$$\frac{P_i}{\sum_I Prod_I^F} + \delta_i^P = \frac{Prod_i}{\sum_i Prod_i} \forall i \in z \qquad \text{Eq. 6.8}$$

$$\frac{A_j}{\sum_J Attr_J^F} + \delta_j^A = \frac{Attr_j}{\sum_j Attr_j} \forall j \in z \qquad \text{Eq. 6.9}$$

$$\sum_{\substack{i \in I \\ j \in J}} \gamma_{ij}^R T_{ij} \geq F_{IJ}^R \forall I, J \in Z \qquad \text{Eq. 6.10}$$

$$\sum_{\substack{i \in I \\ j \in J}} \gamma_{ij}^W T_{ij} \geq F_{IJ}^W \forall I, J \in Z \qquad \text{Eq. 6.11}$$

$$T_{ij} \geq 0 \quad \forall i, j \in z \qquad \text{Eq. 6.12}$$
$$P_i \geq 0 \quad \forall i \in z \qquad \text{Eq. 6.13}$$
$$A_j \geq 0 \quad \forall j \in z \qquad \text{Eq. 6.14}$$

1   The parameters and variables for the SSD-MSD model are explained in Table 1. Note
2 that lower case i and j refer to FAZs whereas upper case I and J refer to FAF regions. In
3 addition, "domestic" flows refer to flows inside the United States (excluding import and export)
4 whereas "total" flows include foreign import and export. Variables δ are auxiliary decision
5 variables and are used to make the formulation more readable.
6
7   **TABLE 1 list of parameters and decision variables in problem 6**

Notation	Description
List of parameters
Z	Set of FAF zones
z	Set of FAZ's
B	Number of FAZ's (215)
Exp_i	Export for FAZ i, $i \in z$
Imp_i	Import for FAZ i, $i \in z$
F_{IJ}	Domestic FAF flow between FAF zones I and J; $I, J \in Z$
$Prod_I^F$	Total production for FAF zone $I \in Z$
$Attr_J^F$	Total attraction for FAF zone $I \in Z$
$Prod_i$	Total production for FAZ i; $i \in z$ obtained from the total generation model
$Attr_i$	Total attraction for FAZ j; $j \in z$ obtained from the total generation model
t_{ij}	Initial trip table between FAZs i and j; $i, j \in z$ obtained from SEMCOD
γ_{ij}^R	Rail availability binary; $i, j \in z$
F_{IJ}^R	Rail flow from FAF zone I to FAF zone J; $I, J \in Z$
γ_{ij}^W	Water mode availability binary; $i, j \in z$
F_{IJ}^W	Water flow from FAF zone I to FAF zone J; $I, J \in Z$
s_{ij}	Priori probability distribution; $i, j \in z$
List of decision variables
P_i	Total production at FAZ$_i$, $i \in z$
A_j	Total attraction at FAZ$_j$, $j \in z$
T_{ij}	Domestic flow from FAZ$_i$ to FAZ$_j$, $i, j \in z$
$\delta^Z{}_{ij}$	Difference between the optimal trip shares and the initial trip shares $\dfrac{t_{ij}}{\sum_{ij} t_{ij}}$; $j \in z$
δ_i^P	Difference between the optimal shares of total production and the initial shares of total production $\dfrac{Prod_i}{\sum_i Prod_i}$; $j \in z$
δ_j^A	Difference between the optimal shares of total attraction and the initial shares of total attraction values $\dfrac{Attr_j}{\sum_j Attr_j}$; $j \in z$

8
9   For the optimization problem (Eq. 6) to be feasible, it is necessary for Eq. 6.15 to hold:

$$\sum_{I \in Z} Prod_I^F - \sum_{i \in z} Exp_i = \sum_{J \in Z} Attr_J^F - \sum_{j \in z} Imp_j = \sum_{I,j} F_{IJ} \qquad \text{Eq. 6.15}$$

10

1   The objective function (Eq. 6.1) minimizes the sum of deviations from initial trip shares.
2   There are three main initial shares involved in CSFFM. The first shares come from the domestic
3   trips estimated by the SEMCOD model (δ_{ij}^Z and T_{ij} are the variables related to this distribution).
4   The second and third shares come from the total productions and attractions estimated by the
5   total generation model. The total productions are represented by P_i and δ_i^P terms and total
6   attractions are represented by A_j and δ_j^A terms. The first term in the objective function has a
7   coefficient of B^2 for scaling purposes.
8   Eqs. 6.2 and 6.3 enforce equivalence of the domestic marginal values. Eq. 6.4 ensures
9   that observed flows between FAF zones are equivalent at the disaggregate level. Eqs. 6.5 and 6.6
10 ensure that the observed FAF total production and attractions are equivalent. Eqs. 6.7, 6.8 and
11 6.9 generate auxiliary decision variables (δ) that show the deviation of the optimal domestic,
12 total production, and total attraction trip shares from their corresponding base trip shares.
13   In the new zoning system, not all the disaggregated zones (FAZs) contain rail yards,
14 seaports, or both. Binary parameters γ_{ij}^R and γ_{ij}^W determine the availability of rail and water
15 modes between zones i and j ($i, j \in z$), respectively. Eqs. 6.10 and 6.11 convey that the
16 proportion of the flow between two FAF zones I and J that moves by the rail or water modes
17 should go through FAZs $i \in I$ and $j \in J$ that have rail or water accessibility between them. Eqs.
18 6.12-6.14 are the non-negativity constrains.
19   Figures 6(a) and 6(d) compare the results of the SSD-MSD and SEMCOD models. The
20 horizontal axes in these figures show the element IDs of the matrices. The element ID for the
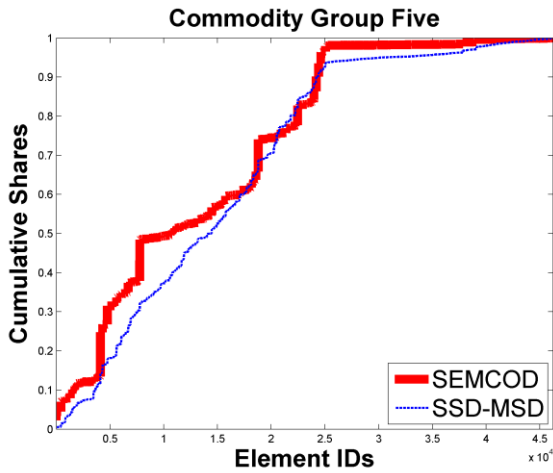21 element C_{mn} of a matrix can be obtained from Eq. 7.
22

$$ID_{mn} = \Sigma_{i=1}^{m-1}\Sigma_{j=1}^{215}1 + \Sigma_{i=1}^{n}1 \qquad\qquad \text{Eq. 7}$$

23

24   The vertical axes of these two figures show the cumulative shares generated by each
25 model. The shares $[s_{ij}]$ for an OD matrix are obtained by dividing the elements of the matrix by
26 the sum of the matrix elements so that values are normalized between zero and one. For an
27 element c_{mn} in the normalized matrix, the cumulative share can be obtained by:
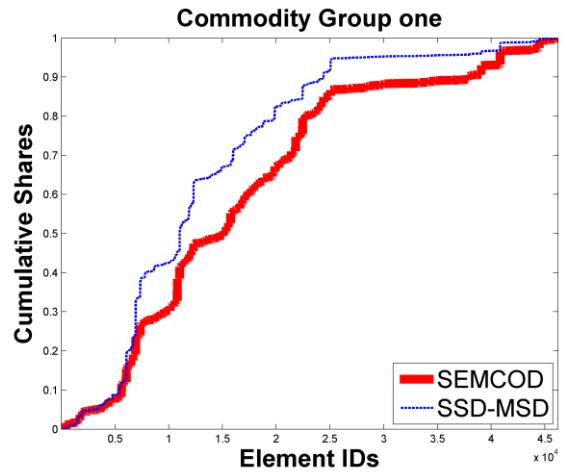28

$$CS_{mn} = \Sigma_{i=1}^{m-1}\Sigma_{j=1}^{215}s_{mn} + \Sigma_{i=1}^{n}s_{mn} \qquad\qquad \text{Eq. 8}$$

29
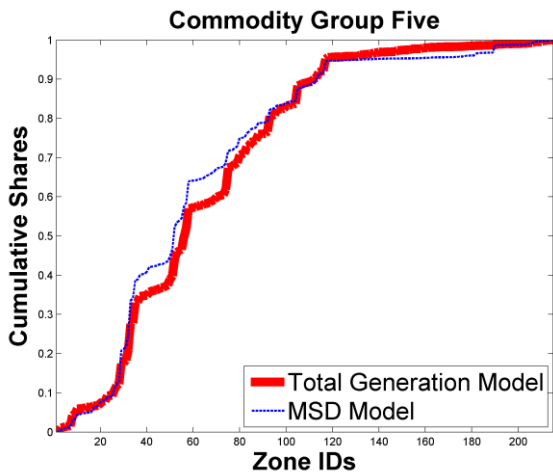
30 **Evaluation of Results**
31   An element-by-element comparison of the cumulative shares of the cells in the base trip
32 table given by the SEMCOD model and the final trip table estimated by SSD-MSD model
33 provides a visual tool to assess the performance of the SSD-MSD model. Ideally, the two trip
34 tables would be the same, and the two graphs corresponding to the cumulative shares of the base
35 and final trip tables would coincide. However, this is not the case in practice due to the presence
36 of constraints. Figures 6(a) and 6(b) show the cumulative shares given by SEMCOD model and
37 those generated by the SSD-MSD. Since SSD-MSD should consider the modal accessibility
38 constraints and consistency with FAF observed flows, the two graphs do not exactly coincide.
39 Therefore there is a gap between them and the spikes in the graphs, which show the share of the
40 corresponding elements, are not exactly similar. However, the results are more promising for
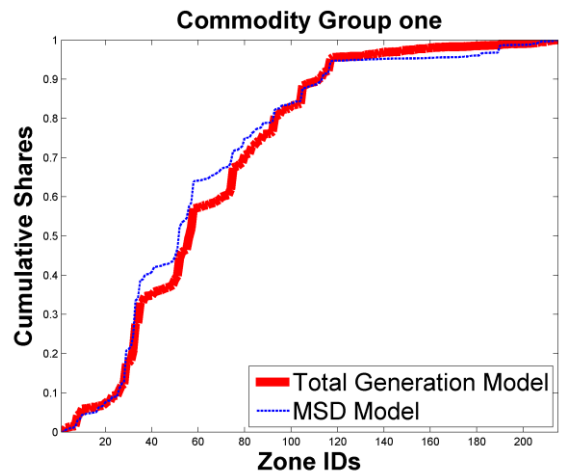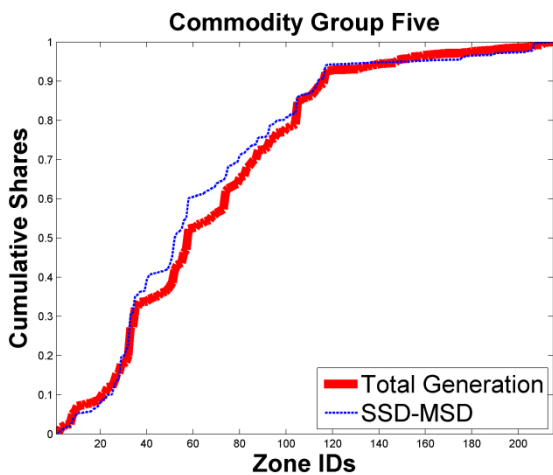41 commodity group five (6(a)) than commodity group one (6(d)).

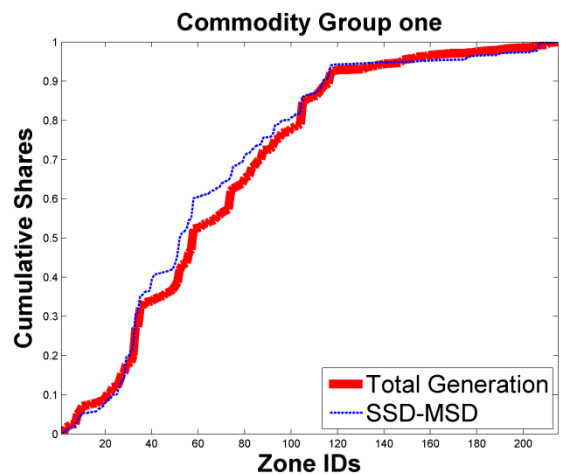**Commodity Group Five**

6(a) Domestic cumulative shares

Commodity Group one

6(d) Domestic cumulative shares

Commodity Group Five

6(b) Total production shares

Commodity Group one

6(e) Total production shares

Commodity Group Five

6(c) Total attraction shares

Commodity Group one

6(f) Total attraction shares

FIGURE 6 Effectiveness of SSD-MSD in maintaining trip shares

1   Figures 6(b), 6(c), 6(e), and 6(f) compare the zonal production and attractions given by
2   the total generation model and those generated by the SSD-MSD. The horizontal axis in these
3   figures show the zone IDs (215 FAZs), and the vertical axis show the cumulative shares
4   generated by the SSD-MSD and total generation models. Here, because there is only one
5   constraint on zonal productions and attractions (matching FAF observations), the cumulative
6   shares generated by the SSD-MSD and total generation models show a smaller gap, and follow a
7   more similar pattern in spikes.
8   Figure 6 implies that SSD-MSD is in general successful in maintaining the initial
9   domestic and total trip shares given by the SEMCOD and total generation models. As also shown
10   in Example 1, MSD-Based Models outperform ME-based models in maintaining the original
11   distribution. The main merit of SSD-MSD model is the possibility to consider all the modal
12   accessibility constraints in generating the final trip table. In addition, the final disaggregated trip
13   table is consistent with aggregate FAF flow tables.
14   CSFFM could not use iterative procedures such as bi-proportional fitting and still address
15   the modal accessibility constraints and be consistent with FAF observed flows. Also the size of
16   the problem prohibited solving an ME-based formulation directly with optimization engines.
17   Therefore, MSD was the only viable solution for the purposes of this study. The contribution of
18   the SSD-MSD in this study was not only in generating more desirable results, but in generating
19   results that could at least satisfy the most basic constraints.

20   **CONCLUSION**
21   A new approach for balancing trip matrices with application in trip distribution and
22   matrix disaggregation has been introduced. The concept of generating the most similar
23   distribution (MSD) instead of Maximum Entropy (ME) principle is the basis of these models.
24   The goal of MSD is to minimize the deviation from the initial trip distribution, considering
25   existing constraints. Two MSD-based objective functions, SSD-MSD and Minimax-MSD were
26   introduced.
27   ME-based formulations are too complex to be directly solved by optimization engines,
28   and the restrictive framework of the iterative procedures used to solve them doesn't allow an
29   exhaustive use of the information in hand. The main advantage of replacing the entropy-based
30   models with any of these new models where appropriate is that the resulting problems can
31   accommodate additional constraints and still be easily solved by optimization engines. In
32   addition, these models could produce more meaningful results than entropy-based models in
33   regional transportation planning studies as demonstrated in Example 1 of this paper.
34   Given the improvements in computational speeds, the authors believe that the theoretical
35   arguments made in this paper and the practical significance of using the alternative models
36   combined implies that these models could be suitable to be used in practice where appropriate.
37   A case study of the California Statewide Freight Forecasting Model (CSFFM) was
38   presented to show the merits of using MSD-based formulations. In this study balancing factors
39   for 3 initial trip shares pertaining to the domestic flow, total production, and total consumption at
40   a fine zonal level were estimated, while respecting the observed flows between the more
41   aggregate FAF zones, and satisfying the modal accessibility constraints. These goals could not
42   be addressed in conventional bi-proportional matrix fitting techniques such as the Furness
43   method. In addition, the size of the problem prohibited solving an ME-based formulation directly
44   with optimization software. Hence, the MSD-based models were determined to be the only
45   viable options in this study. Some of the attractive features of using the SSD-MSD model for

CSFFM were demonstrated by generating and comparing graphs of the cumulative base and optimal shares.

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